

# Equivariant Homotopy Theory

We will always assume that  $G$  is finite. All cats are  $\infty$ -cats.

Motivation spectral Mackey functor is a model of equivariant htpy thy.

- History : Guillon - May 2013 : model of  $G$ -spectra  
Barnick - Glasman - Shah 2015 : model of equivariant algebraic  $K$  for Waldhausen  $\infty$ -cat w/  $G$ -action

Barnick 2017 : model of equivariant htpy thy extended (come earlier) to profinite  $G$ .

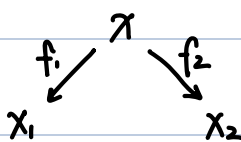
## PART I Equivariant (stable) htpy via spectral Mackey functor.

- Spectral Mackey functor.

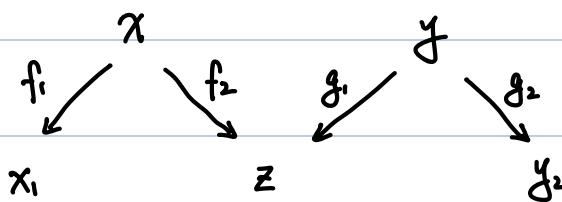
$\mathcal{C}$  cat w/ pullbacks. Span  $\mathcal{C}$  = cat of spans in  $\mathcal{C}$ .

Span  $\mathcal{C}$  obj : obj  $\mathcal{C}$

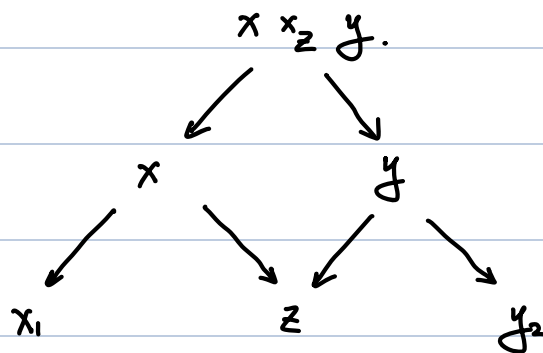
mor :  $f_i \in \text{Mor } \mathcal{C}$ .



compositions are given by pullback. i.e.



→ span from  $x_1$  to  $y$



Recall that, for  $\mathcal{C} = \text{Fin}_G = \text{cat of finite } G\text{-sets}$ , after Grothendieck completion on each morphism in  $\text{Span}(\text{Fin}_G)$ , one gets the Burnside category  $A(G) = \text{Span}^+(\text{Fin}_G)$ .

Rk In Barrick's paper, he defined the effective Burnside cat  $A^{\text{eff}}(G) := \text{Span}(\text{Fin}_G)$ .

- Omitted

X Prop (Barrick 17, Prop 4.3) If  $\mathcal{C}$  has

- disjunctive
- ① pullbacks.
  - ② finite coproducts.
  - ③ finite coproducts are disjoint & universal, i.e.

$$\prod_{i \in I} \mathcal{C}/X_i \xrightarrow{\cong} \mathcal{C}/\coprod_{i \in I} X_i$$

then  $\text{Span } \mathcal{C}$  has direct sums, i.e.

- semi-additive
- 1)  $\text{Span } \mathcal{C}$  is pointed
  - 2)  $\text{Span } \mathcal{C}$  has finite products & coproducts
  - 3) For  $I$  finite,  $X_i \in \text{Span } \mathcal{C}$ ,  $i \in I$ .

$$\coprod_{i \in I} X_i \xrightarrow{\cong} \prod_{i \in I} X_i$$

Prop  $\text{Span } \mathcal{C}$  has a symmetric monoidal structure w/ unit  $*$ .

Moreover,  $\mathcal{C} \mapsto \text{Span } \mathcal{C}$  is lax sym. mon. when the source is cartesian monoidal.

IDEA. Induced by products on  $\mathcal{C}$ .

Barwick 2017 Theorem 2.15.

Rk  $\text{Span } \mathcal{C}$  is not additive.

If  $\mathcal{C}$  semi-additive, then  $\text{CMon}(\mathcal{C}) \cong \mathcal{C}$ .

Def Let  $\mathcal{C}$  be disjointive.  $\mathcal{D}$  be semi-additive.  $\mathcal{D}$ -valued Mackey functor on  $\mathcal{C}$  is the (co)product-preserving functor  $\text{Span } \mathcal{C} \rightarrow \mathcal{D}$ .

(has pullbacks / finite coproducts + disjoint) (prod. finite products & coproducts & composites)

Write  $\text{Mack}(\mathcal{C}; \mathcal{D}) := \overline{\text{Fun}}^{\oplus}(\text{Span } \mathcal{C}, \mathcal{D})$

Def  $\text{Mack}_G(\mathcal{S}_p) = \text{Mack}(\text{Fin}_G; \mathcal{S}_p)$  is called the cat of spectral Mackey functors. Denoted by  $\text{Sp}_G$ .

Prop 1) For  $\mathcal{D}$  semi-additive / additive / stable,  $\text{Mack}(\mathcal{C}; \mathcal{D})$  also semi-additive / additive / stable.

2)  $\text{Mack}(\mathcal{C}; \mathcal{D})$  admits a sym. mon. structure by Day convolution, if  $\mathcal{D}$  has a sym. mon. structure.

In this case, sym. mon. str. is given by the left

Kan extension:

$$\begin{array}{ccc}
 \mathcal{C} \times \mathcal{C} & \xrightarrow{(F, G)} & \mathcal{D} \times \mathcal{D} \\
 \otimes_e \downarrow & & \downarrow \otimes_{\mathcal{D}} \\
 \mathcal{C} & \xrightarrow{F \otimes G} & \mathcal{D}
 \end{array}$$

$$(F \otimes G)(c) = (c_1 \times c_2 \rightarrow c) \in \mathcal{C} \xrightarrow{\text{colim}} \mathcal{C} \times \mathcal{C} \xrightarrow{F(c_1) \otimes G(c_2)}$$

- Equivariant (stable) homotopy theory.

Theorem ( Guillou - May )  $\text{Mack}_G(\text{Sp}) = \text{Sp}_G$  is equivalent to the cat of genuine  $G$ -spectra ( orthogonal  $G$ -spectra model )

Now to set up the basic equivariant hopy thry via spectral Mackey functor.

- Unstable settings.

Def cat of  $G$ -spaces  $S_G := \text{Fun}(\mathcal{O}_G^{\text{op}}, S)$ .

.. pred ..  $S_{G,*} := \text{Fun}(\mathcal{O}_G^{\text{op}}, S_*)$

Here  $\mathcal{O}_G =$  orbit cat

$S =$  cat of spaces.

To get a  $G$ -space from a  $G$ -spectrum, one needs the infinite loop space functor.

Def infinite loop space functor  $\Omega^\infty$  is the composite

$$\text{Sp}_G = \text{Fun}^\oplus(\text{Span}(\text{Fin}_G), \text{Sp}) \xrightarrow{\Omega^\infty} \text{Fun}^x(\text{Span}(\text{Fin}_G), S_*)$$

$$\rightarrow \text{Fun}^x(\text{Fin}_G^{\text{op}}, S_*)$$

$$\cong \text{Fun}(\mathcal{O}_G^{\text{op}}, S_*) = S_{G,*}.$$

abuse the notation.

It has a left adjoint  $\Sigma_+^\infty : \mathcal{S}a.* \rightarrow \mathcal{S}pa.$

Lem Let  $I \in \text{Fin}_G$ .  $X \in \mathcal{S}pa.$

$$\text{Map}_G(\Sigma_+^\infty I, X) = X(I)$$

mapping spectrum

- Fixed pts, restrictions, inductions.

Def Let  $X \in \mathcal{S}pa.$  the categorical fixed pts of  $X$  on  $K \leq G$  is

$$X^K := X(G/K) = \text{Map}_G(\Sigma_+^\infty G/K, X).$$

Def Let  $K \leq G.$  We have

restriction map  $\text{res}_K^G : \mathcal{S}pa \rightarrow \mathcal{S}p_K$

induction map  $\text{ind}_K^G : \mathcal{S}p_K \rightarrow \mathcal{S}pa$

s.t. 1) By the following adjoint pair.

$$\begin{array}{ccc} \text{Fin}_G & & \\ \text{Forget} \downarrow & \searrow & \\ & G \times_K (-) & \rightarrow \mathcal{S}p \\ \text{Fin}_K & \nearrow & \end{array}$$

one has

$$\text{res}_K^G(X) = X \circ (G \times_K (-))$$

$$\text{ind}_K^G(X) = X \circ \text{Forget}.$$

2)  $\text{res}_K^G$  and  $\text{ind}_K^G$  both preserve pullback.

$$\text{ind}_K^G \dashv \text{res}_K^G.$$

3) (Wittmüller iso)  $\text{res}_K^G \dashv \text{ind}_K^G$ .

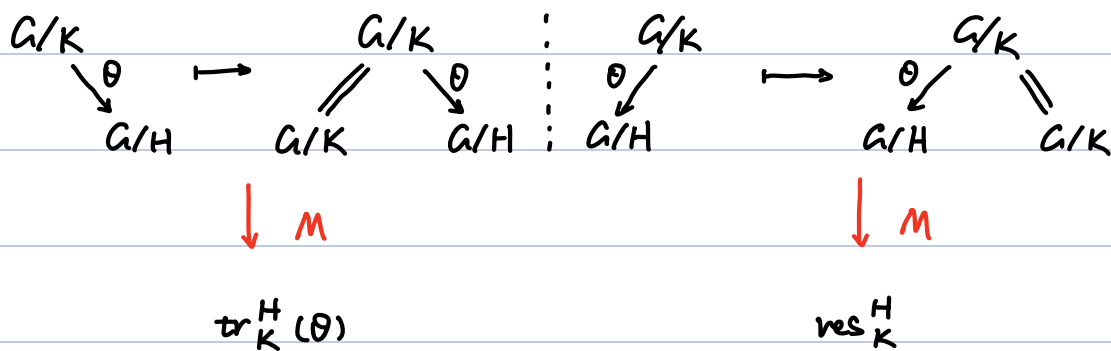
e.g.  $\text{ind}_K^G(X)(G/e) = (X \circ \text{Forget})(G/e)$   
 $= X(\coprod_{g \in G/K} gK)$   
 $\cong X(\coprod_{g \in G/K} K)$   
 $= \bigoplus_{g \in G/K} X(K/e)$

i.e.  $\text{ind}_K^G X = |G/K|$ -fold coproduct of  $X$ .

e.g. Traditionally, the Mackey functor  $M: \text{Span}(\text{Fin}_G) \rightarrow \text{Ab}$  consists of following data:

1) sending  $\sqcup$  to  $\bigoplus$ , i.e.  $M(\coprod G/K) = \bigoplus M(G/K)$ .

2) since  $\mathcal{O}_G \rightarrow \text{Span}(\text{Fin}_G)$  ⋮  $\mathcal{O}_G^{\text{op}} \rightarrow \text{Span}(\text{Fin}_G)$



for  $K \leq H \leq G$ .  $\theta: gK \mapsto gH$ .

s.t.  $\text{tr}$ ,  $\text{res}$  satisfy chain rules, and

$\forall \gamma \in W_{HK}$ ,  $x \in M(G/K)$ ,  $y \in M(G/H)$

①  $\text{tr}_K^H(\gamma \cdot x) = \text{tr}_K^H(x)$

②  $\gamma \cdot \text{res}_K^H(x) = \text{res}_K^H(x)$

③  $\text{res}_K^H \text{tr}_K^J(z) = \sum_{\gamma} \gamma \cdot \text{tr}_{J \cap K}^K(x)$ ,  $J, K \leq H$ .

In our case.  $G = C_p$ . only two orbits  $C_p/C_p$  and  $C_p/e$ . So we have

$$\text{tr}_e^{C_p} : M(C_p/e) \rightarrow M(C_p/C_p)$$

$$\text{res}_e^{C_p} : M(C_p/C_p) \rightarrow M(C_p/e)$$

To compute the Day convolution of  $M$  &  $N$ .

$$(M \otimes N)(C_p/C_p)$$

$$\text{tr}_e^{C_p} \Big| \Big| \text{res}_e^{C_p}$$

$$(M \otimes N)(C_p/e)$$

Note that by definition

$$(M \otimes N)(x) = \text{colim}_{x_1 \times x_2 \rightarrow x} M(x_1) \otimes N(x_2)$$

$$= \text{colim}_{e/(x_1 \times x_2 \rightarrow x)} M(x_1) \otimes N(x_2)$$

In our case. it's

$$(M \otimes N)(C_p/e) = \text{colim}_{A \times B \rightarrow C_p/e} M(A) \otimes N(B).$$

$$= \text{evaluate at final obj of } \text{Fina}_{A \times B} \downarrow C_p/e$$

$$= M(C_p/e) \otimes N(C_p/e).$$

$(M \otimes N)(C_p/C_p)$  is a little bit tricky. since we need to have a well-defined transfer satisfying the desired properties.

- Artificially introduce all transfers.

$$(M \otimes N)(C_p/C_p) = (M(C_p/C_p) \otimes N(C_p/C_p)) \oplus \text{im}(\text{tr})$$

$$\text{where } \text{im} \text{ tr} = (M(C_p/e) \otimes N(C_p/e)) / C_p$$

s.t.  $\text{tr}(\gamma(a \otimes b)) = \text{tr}(a \otimes b)$ , as in ①.

$\gamma \in G_p$  acts diagonally  $\rightsquigarrow$  compatible w/ im tr.

Thus we have

Here  $r(x) = x$   
is the fixed pt

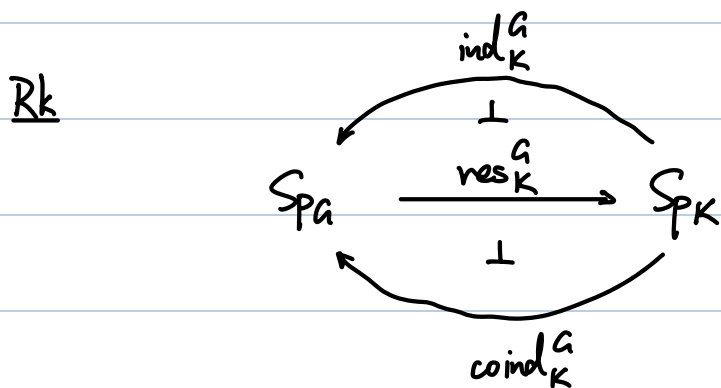
$$\begin{array}{ccc}
 & M(C_p/e) \otimes M(C_p/e) & \\
 & \uparrow & \downarrow \text{tr}_e^{C_p} \\
 \text{res}_e^{C_p}(x \otimes y) & & \text{s.t. } \text{tr}_e^{C_p}(\gamma \cdot x) = \text{tr}_e^{C_p}(x) \\
 = r_M(x) \otimes r_N(y) & & x \otimes \text{tr}(y) \sim \text{tr}(r(x) \otimes y) \\
 & & \text{tr } x \otimes y \sim \text{tr}(x \otimes r(y))
 \end{array}$$

$$[ (M(C_p/C_p) \otimes N(C_p/C_p)) \oplus \text{im tr} ] / \sim$$

## PART II HHR norms.

We want to discuss smash products where the index set admits a group action. We need to interpolate between  $S_{pK}$  ranging over subgps  $K \leq G$ .

Def coinduction map  $\text{coind}_K^G : S_{pK} \rightarrow S_{pG}$  is given by  $\text{coind}_K^G(X) = \text{Map}_K(G, X)$



Wirthmüller iso :  $\text{ind}_K^G \cong \text{coind}_K^G$ .

Rk Can also write  $\text{coind}_K^G : S_K \rightarrow S_G$ .  $X \mapsto \text{Map}_K(G, X)$  mapping space



Construction  $N_K^G$  is the sym. mon. left Kan extension as follows:

$$\begin{array}{ccc} S_K & \xrightarrow{\text{coind}_K^G} & S_G \\ (-)_+ \downarrow & & \downarrow (-)_+ \\ S_{K,*} & \xrightarrow{N_K^G} & S_{G,*} \end{array}$$

The HHR norm is defined to be the "stable version":

$$\begin{array}{ccc} S_{K,*} & \xrightarrow{N_K^G} & S_{G,*} \\ \Sigma_+^\infty \downarrow & & \downarrow \Sigma_+^\infty \\ S_{pK} & \xrightarrow{N_K^G} & S_{pG} \end{array}$$

Here we abuse the notation.

Cor  $\text{res}_e^G N_e^G \cong (-)^{\otimes |G|}$ .

Aside Traditionally,  $X$  orthogonal spectra.  $X^{an} \supseteq C_n$  on factors.  
 $K \leq G$ .  $X$   $K$ -spectrum, then HHR norm is defined to be

$$N_K^G X := \bigwedge_{g:K \in G/K} (g:K) \wedge_H X \cong \bigwedge^{|G/K|} X$$

### PART III Isotropy separation

Denote  $S_{pG}^{\mathcal{P}}$  = full subcat of  $S_{pG}$  gen. by orbits  $\Sigma_+^\infty G/K$  for  $K \leq G$ .

Def / Thm The geometric fixed pts functor is defined to be

the localization  $\Phi^G : S_{pa} \rightarrow S_{pa} / S_{pa}^{\mathcal{P}} \cong S_p$   
 where the latter is an equiv between sym. mon. cats.

Now to relate this def. to the one we're familiar with.

Let  $\mathcal{F}$  = family of subgps of  $G$  closed under conjugation.

$\mathcal{O}_a^{\mathcal{F}} \subseteq \mathcal{O}_a$  be the full subcat spanned by transitive  $G$ -sets whose isotropy lies in  $\mathcal{F}$

Aside isotropy  $\approx$  stabilizer

$$E\mathcal{F} := \operatorname{colim}_{\mathcal{O}_a^{\mathcal{F}}} (\mathcal{O}_a^{\mathcal{F}} \rightarrow S_G) \quad \text{Yoneda emb + restriction.}$$

$$= \operatorname{colim}_{G/H \in \mathcal{O}_a^{\mathcal{F}}} (\operatorname{Hom}_{S_G}(G/H, -)).$$

$$\underline{\text{Prop}} \quad (E\mathcal{F})^K \cong \begin{cases} * & K < G \\ \emptyset & K = G. \end{cases}$$

So  $E\mathcal{F}$  is a  $G$ -space.

$$- (E\mathcal{F})^K = (E\mathcal{F})(G/K) = \operatorname{colim}_{G/H \in \mathcal{O}_a^{\mathcal{F}}} \operatorname{Hom}_{S_G}(G/H, G/K).$$

Construction Let  $\widetilde{E\mathcal{F}} = \operatorname{cofib}(E\mathcal{F}_+ \rightarrow * \cong S^0)$ . We get

the isotropy separation map

$$E\mathcal{F}_+ \rightarrow S^0 \rightarrow \widetilde{E\mathcal{F}}$$

$$\text{and } \widetilde{E\mathcal{F}}^K \cong \begin{cases} * & K < G \\ S^0 & K = G \end{cases}$$

Thm  $\Phi^G \simeq (\widetilde{EF} \otimes -)^G$

By def.  $\Phi^G$  is sym. mon.

Prop  $\Phi^G \circ \Sigma_+^\infty \simeq \Sigma_+^\infty \circ (-)^G$  as sym. mon.

Some other relations:

Thm  $(-)^G$  is initial among  $\text{Fun}_{\text{colim}}^{\text{l.s.m}}(\text{Spa}, \text{Sp})$  the colimit preserving lax sym. mon. functors.

$\Rightarrow \exists!$  lax sym. mon. transformation  $(-)^G \rightarrow \Phi^G$ .

- Homotopy fixed pts

FACT  $E$  semi-additive cat. then

$$\text{Mack}(\text{Fin}_G^{\text{free}}; E) \simeq \text{Fun}(BG, E)$$

cat of free finite  $G$ -sets.

Let  $X \in \text{Spa}$ .  $K \leq G$ . then

$$X^{hK} := \text{Map}(BG, X)^K$$

$$\begin{array}{ccc} \text{Span}(\text{Fin}_G^{\text{free}}) & \longrightarrow & \text{Span}(\text{Fin}_G) \\ & \searrow & \downarrow \\ \text{Mack}(\text{Fin}_G^{\text{free}}; E) & & \text{Mack}(\text{Fin}_G; E) \\ \cong & & \downarrow \\ \text{Fun}(BG, E) & & E \end{array}$$