Equivariant Homotopy Theory

We will always assume that G is finite. All cats are ∞-cats.

Motivation spectral Mackey functor is a model of equivariant htpy thy.

- History: Guillou – May 2013: model of G-spectra

  Barwick – Glasman – Shah 2015: model of equivariant algebraic K for Waldhausen ∞-cat w/ G-action

  Barwick 2017: model of equivariant htpy thy extended to profinite G.
  (come earlier)

PART I Equivariant (stable) htpy via spectral Mackey functor.

- Spectral Mackey functor.

  $\mathcal{C}$ cat w/ pullbacks. Span $\mathcal{C}$ = cat of spans in $\mathcal{C}$.

  Span $\mathcal{C}$ obj : obj $\mathcal{C}$

  mor :
$$\begin{tikzcd} & x \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 & & X_2 \end{tikzcd}$$
  . $f_i \in \mathrm{Mor}\,\mathcal{C}$.

  compositions are given by pullback. i.e.

$$\begin{tikzcd} & x \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & y \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ X_1 & & z & & y_2 \end{tikzcd}$$

$\rightsquigarrow$ span from x_1 to y

$$\begin{tikzcd}[column sep=small]
& & x \times_z y \arrow[dl] \arrow[dr] & & \\
& x \arrow[dl] \arrow[dr] & & y \arrow[dl] \arrow[dr] & \\
x_1 & & z & & y_2
\end{tikzcd}$$

Recall that, for $\mathcal{C} = \mathrm{Fin}_G$ = cat of finite G-sets, after Grothendieck completion on each morphism in $\mathrm{Span}(\mathrm{Fin}_G)$, one gets the Burnside category $A(G) = \mathrm{Span}^+(\mathrm{Fin}_G)$.

Rk In Barwick's paper, he defined the effective Burnside cat $A^{\mathrm{eff}}(G) := \mathrm{Span}(\mathrm{Fin}_G)$.

\- Omitted

X Prop (Barwick 17, Prp 4.3) If $\mathcal{C}$ has

disjunctive
① pullbacks.
② finite coproducts.
③ finite coproducts are disjoint & universal, i.e.

$$\prod_{i\in I} \mathcal{C}_{/X_i} \xrightarrow{\cong} \mathcal{C}_{/\coprod_{i\in I} X_i}$$

then $\mathrm{Span}\,\mathcal{C}$ has direct sums, i.e.

semi-additive
1) $\mathrm{Span}\,\mathcal{C}$ is pointed
2) $\mathrm{Span}\,\mathcal{C}$ has finite products & coproducts
3) For I finite, $X_i \in \mathrm{Span}\,\mathcal{C}$, $i \in I$,

$$\coprod_{i\in I} X_i \xrightarrow{\cong} \prod_{i\in I} X_i$$

Prop $\mathrm{Span}\,\mathcal{C}$ has a symmetric monoidal structure w/ unit $*$.
Moreover, $\mathcal{C} \longmapsto \mathrm{Span}\,\mathcal{C}$ is lax sym. mon. when the source is cartesian monoidal.

IDEA. Induced by products on $\mathcal{C}$.
Barwick 2017 Theorem 2.15.

Rk $\mathrm{Span}\,\mathcal{C}$ is not additive.
If $\mathcal{C}$ semi-additive, then $\mathrm{CMon}(\mathcal{C}) \simeq \mathcal{C}$.

Def Let $\mathcal{C}$ be disjunctive (has pullbacks / finite coproducts + disjoint). $\mathcal{D}$ be semi-additive (pted. finite products & coprods & commute).
$\mathcal{D}$-valued Mackey functor on $\mathcal{C}$ is the (co)product-preserving functor $\mathrm{Span}\,\mathcal{C} \to \mathcal{D}$.
Write $\mathrm{Mack}(\mathcal{C};\mathcal{D}) := \mathrm{Fun}^{\oplus}(\mathrm{Span}\,\mathcal{C}, \mathcal{D})$

Def $\mathrm{Mack}_G(\mathrm{Sp}) = \mathrm{Mack}(\mathrm{Fin}_G; \mathrm{Sp})$ is called the cat of spectral Mackey functors. Denoted by Sp_G.

Prop 1) For $\mathcal{D}$ semi-additive / additive / stable. $\mathrm{Mack}(\mathcal{C};\mathcal{D})$ also semi-additive / additive / stable.

2) $\mathrm{Mack}(\mathcal{C};\mathcal{D})$ admits a sym. mon. structure by Day convolution, if $\mathcal{D}$ has a sym. mon. structure.
In this case. sym. mon. str. is given by the left

Kan extension :

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "(F.G)"] \arrow[d, "\otimes_{\mathcal{C}}"'] & \mathcal{D} \times \mathcal{D} \arrow[d, "\otimes_{\mathcal{D}}"] \\ \mathcal{C} \arrow[r, dashed, "F \otimes G"'] & \mathcal{D} \end{tikzcd}$$

$$(F \otimes G)(c) = \operatorname*{colim}_{(c_1 \times c_2 \to c) \in \mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{C} \times \mathcal{C}} F(c_1) \otimes_{\mathcal{D}} G(c_2)$$

- Equivariant (stable) homotopy theory.

<u>Theorem</u> (Guillou - May) $Mack_G(Sp) = Sp_G$ is equivalent to the cat of genuine G-spectra (orthogonal G-spectra model)

Now to set up the basic equivariant htpy thy via spectral Mackey functor.

- Unstable settings.

<u>Def</u> cat of G-spaces $\mathcal{S}_G := \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S})$.

" pted " $\mathcal{S}_{G,*} := \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S}_*)$

Here $\mathcal{O}_G$ = orbit cat

$\mathcal{S}$ = cat of spaces.

To get a G-space from a G-spectrum, one needs the infinite loop space functor.

<u>Def</u> infinite loop space functor Ω^∞ is the composite

$$Sp_G = \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathrm{Fin}_G), Sp) \xrightarrow{\Omega^\infty} \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}_G), \mathcal{S}_*)$$

abuse the notation.

$$\to \mathrm{Fun}^{\times}(\mathrm{Fin}_G^{op}, \mathcal{S}_*)$$

$$\simeq \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S}_*) = \mathcal{S}_{G,*}.$$

It has a left adjoint $\Sigma^\infty_+ : S_{G,*} \longrightarrow Sp_G$.

<u>Lem</u> Let $I \in Fin_G$. $X \in Sp_G$.

$$\mathrm{Map}_G(\Sigma^\infty_+ I, X) = X(I)$$

mapping spectrum

- Fixed pts. restrictions, inductions.

<u>Def</u> Let $X \in Sp_G$. the categorical fixed pts of X on $K \leq G$ is

$$X^K := X(G/K) = \mathrm{Map}_G(\Sigma^\infty_+ G/K, X).$$

<u>Def</u> Let $K \leq G$. We have

restriction map $\quad res^G_K : Sp_G \longrightarrow Sp_K$

induction map $\quad ind^G_K : Sp_K \longrightarrow Sp_G$

s.t. 1) By the following adjoint pair.

$$\begin{tikzcd}
Fin_G \arrow[d, shift right, "Forget"'] \arrow[dr] & \\
Fin_K \arrow[u, shift right, "G\times_K(-)"'] \arrow[r] & Sp
\end{tikzcd}$$

one has

$$res^G_K(X) = X \circ (G \times_K (-))$$

$$ind^G_K(X) = X \circ \mathrm{Forget}.$$

2) res^G_K and ind^G_K both preserve pullback.

$$ind^G_K \dashv res^G_K.$$

3) (Wirthmüller iso) $\quad \mathrm{res}^G_K \dashv \mathrm{ind}^G_K$.

e.g.
$$\begin{aligned}
\mathrm{ind}^G_K(X)(G/e) &= (X \circ \mathrm{Forget})(G/e) \\
&= X\Big(\coprod_{g \in G/K} gK\Big) \\
&\cong X\Big(\coprod_{g \in G/K} K\Big) \\
&= \bigoplus_{g \in G/K} X(K/e)
\end{aligned}$$

i.e. $\mathrm{ind}^G_K X = |G/K|$-fold coproduct of X.

e.g. Traditionally, the Mackey functor $M : \mathrm{Span}(\mathrm{Fin}_G) \longrightarrow \mathrm{Ab}$ consists of following data:

1) sending $\coprod$ to $\oplus$, i.e. $M(\coprod G/K) = \oplus M(G/K)$.

2) since $\mathcal{O}_G \to \mathrm{Span}(\mathrm{Fin}_G)$ $\quad\vdots\quad$ $\mathcal{O}_G^{op} \longrightarrow \mathrm{Span}(\mathrm{Fin}_G)$

$$\begin{tikzcd}
G/K \arrow[dr, "\theta"] & \arrow[r, mapsto] & & G/K \arrow[dl, equal] \arrow[dr, "\theta"] & \\
& G/H & G/K & & G/H
\end{tikzcd}
\qquad \vdots \qquad
\begin{tikzcd}
& G/K \arrow[dl, "\theta"'] & \arrow[r, mapsto] & & G/K \arrow[dl, "\theta"'] \arrow[dr, equal] & \\
G/H & & & G/H & & G/K
\end{tikzcd}$$

$$\Big\downarrow M \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow M$$

$$\mathrm{tr}^H_K(\theta) \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{res}^H_K$$

for $K \leq H \leq G$. $\theta : gK \mapsto gH$.

s.t. tr, res satisfy chain rules, and

$\forall\ \gamma \in W_H K$, $x \in M(G/K)$, $y \in M(G/H)$

① $\mathrm{tr}^H_K(\gamma \cdot x) = \mathrm{tr}^H_K(x)$

② $\gamma \cdot \mathrm{res}^H_K(x) = \mathrm{res}^H_K(x)$

③ $\mathrm{res}^H_K\, \mathrm{tr}^J_K(z) = \sum_\gamma \gamma \cdot \mathrm{tr}^K_{J \cap K}(x)$, $\quad J, K \leq H$.

In our case. $G = C_p$. only two orbits C_p/C_p and C_p/e. So we have

$$\mathrm{tr}_e^{C_p} : M(C_p/e) \longrightarrow M(C_p/C_p)$$

$$\mathrm{res}_e^{C_p} : M(C_p/C_p) \longrightarrow M(C_p/e)$$

To compute the Day convolution of M & N.

$$\begin{tikzcd} (M \otimes N)(C_p/C_p) \arrow[d, shift left, "\mathrm{res}_e^{C_p}"] \\ (M \otimes N)(C_p/e) \arrow[u, shift left, "\mathrm{tr}_e^{C_p}"] \end{tikzcd}$$

Note that by definition

$$(M \otimes N)(x) = \operatorname*{colim}_{x_1 \times x_2 \to x} M(x_1) \otimes N(x_2)$$

$$= \operatorname*{colim}_{\mathcal{C}/(x_1 \times x_2 \to x)} M(x_1) \otimes N(x_2)$$

In our case. it's

$$(M \otimes N)(C_p/e) = \operatorname*{colim}_{A \times B \to C_p/e} M(A) \otimes N(B).$$

$= \text{evaluate at final obj of } \mathrm{Fin}_G/\begin{smallmatrix} A \times B \\ \downarrow \\ C_p/e \end{smallmatrix}$

$$= M(C_p/e) \otimes N(C_p/e).$$

$(M \otimes N)(C_p/C_p)$ is a little bit tricky. since we need to have a well-defined transfer satisfying the desired properties.

- Artificially introduce all transfers.

$$(M \otimes N)(C_p/C_p) = (M(C_p/C_p) \otimes N(C_p/C_p)) \oplus \mathrm{im}(\mathrm{tr})$$

where $\mathrm{im}\,\mathrm{tr} = (M(C_p/e) \otimes N(C_p/e))/C_p$

s.t. $\quad tr(\gamma(a \otimes b)) = tr(a \otimes b)$, as in ①.

$\gamma \in C_p$ acts diagonally $\rightsquigarrow$ compatible w/ im tr.

Thus we have

$\boxed{\text{Here } r(x) = x \text{ is the fixed pt}}$

$$\begin{tikzcd}
M(C_p/e) \otimes M(C_p/e) \arrow[d, shift left, "tr_e^{C_p}"] \\
\left[\left(M(C_p/C_p) \otimes N(C_p/C_p) \right) \oplus \text{im } tr \right] / \sim \arrow[u, shift left, "{\substack{res_e^{C_p}(x \otimes y) \\ = r_M(x) \otimes r_N(y)}}"]
\end{tikzcd}$$

s.t. $\quad tr_e^{C_p}(\gamma \cdot x) = tr_e^{C_p}(x)$

$x \otimes tr(y) \sim tr(r(x) \otimes y)$

$tr x \otimes y \sim tr(x \otimes r(y))$

PART II HHR norms.

We want to discuss smash products where the index set admits a group action. We need to interpolate between Sp_K ranging over subgps $K \leq G$.

Def coinduction map $\quad coind_K^G : Sp_K \longrightarrow Sp_G$ is given by $\quad coind_K^G(X) = Map_K(G, X)$

Rk

$$\begin{tikzcd}
Sp_G \arrow[rr, "res_K^G"] & & Sp_K \arrow[ll, bend right=50, "ind_K^G"', "\perp"] \arrow[ll, bend left=50, "coind_K^G", "\perp"']
\end{tikzcd}$$

Wirthmüller iso : $\quad ind_K^G \simeq coind_K^G$.

Rk Can also write $\quad coind_K^G : S_K \longrightarrow S_G, \quad X \longmapsto \underset{\text{mapping space}}{Map_K}(G, X)$

Construction N_K^G is the sym. mon. left Kan extension as follows:

$$\begin{tikzcd} \mathcal{S}_K \arrow[r, "\mathrm{coind}_K^G"] \arrow[d, "(-)_+"'] & \mathcal{S}_G \arrow[d, "(-)_+"] \\ \mathcal{S}_{K,*} \arrow[r, dashed, "N_K^G"] & \mathcal{S}_{G,*} \end{tikzcd}$$

The HHR norm is defined to be the "stable version":

$$\begin{tikzcd} \mathcal{S}_{K,*} \arrow[r, "N_K^G"] \arrow[d, "\Sigma_+^\infty"'] & \mathcal{S}_{G,*} \arrow[d, "\Sigma_+^\infty"] \\ \mathrm{Sp}_K \arrow[r, dashed, "N_K^G"] & \mathrm{Sp}_G \end{tikzcd}$$

Here we abuse the notation.

Cor $\mathrm{res}_e^G N_e^G \simeq (-)^{\otimes |G|}$.

Aside Traditionally, X orthogonal spectra. $X^{\wedge n} \circlearrowleft C_n$ on factors. $K \leq G$. X K-spectrum, then HHR norm is defind to be

$$N_K^G X := \bigwedge_{g_i K \in G/K} (g_i K) \wedge_H X \cong \bigwedge^{|G/K|} X$$

PART III Isotropy separation

Denote Sp_G^φ = full subcat of Sp_G gen. by orbits $\Sigma_+^\infty G/K$ for $K \leq G$.

Def / Thm The geometric fixed pts functor is defined to be the localization $\Phi^G : Sp_G \xrightarrow{\text{localize}} Sp_G / Sp_G^{\varphi} \simeq Sp$

where the latter is an equiv between sym. mon. cats.

Now to relate this def. to the one we've familiar with.

Let $\mathcal{F}$ = family of subgps of G closed under conjugation.

$\mathcal{O}_G^{\mathcal{F}} \subseteq \mathcal{O}_G$ be the full subcat spanned by transitive G-sets whose isotropy lies in $\mathcal{F}$

Aside isotropy $\approx$ stabilizer

$$E\mathcal{F} := \operatorname{colim}_{\mathcal{O}_G^{\mathcal{F}}} \left(\mathcal{O}_G^{\mathcal{F}} \xrightarrow{\text{Yoneda emb + restriction.}} S_G \right)$$

$$= \operatorname{colim}_{G/H \in \mathcal{O}_G^{\mathcal{F}}} \left(\mathrm{Hom}_{S_G}(G/H , -) \right).$$

Prop $(E\mathcal{F})^K \simeq \begin{cases} * , & K < G \\ \phi , & K = G. \end{cases}$

So $E\mathcal{F}$ is a G-space.

- $(E\mathcal{F})^K = (E\mathcal{F})(G/K) = \operatorname{colim}_{G/H \in \mathcal{O}_G^{\mathcal{F}}} \mathrm{Hom}_{S_G}(G/H , G/K).$

Construction Let $\widetilde{E\mathcal{F}} = \operatorname{cofib}(E\mathcal{F}_+ \longrightarrow *_+ \simeq S^0)$. We get the isotropy separation map

$$E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}}$$

and $\widetilde{E\mathcal{F}}^K \simeq \begin{cases} * , & K < G \\ S^0 , & K = G \end{cases}$

Thm $\Phi^G \simeq (\widetilde{EF} \otimes -)^G$

By def. Φ^G is sym. mon.

Prp $\Phi^G \circ \Sigma^\infty_+ \simeq \Sigma^\infty_+ \circ (-)^G$ as sym. mon.

Some other relations:

Thm $(-)^G$ is initial among $\mathrm{Fun}^{l.s.m}_{colim}(Sp_G, Sp)$, the colimit preserving lax sym. mon. functors.

$\Rightarrow$ $\exists!$ lax sym. mon. transformation $(-)^G \longrightarrow \Phi^G$.

- Homotopy fixed pts

FACT E semi-additive cat, then

cat of free finite G-sets.

$$\mathrm{Mack}(\mathrm{Fin}_G^{free} ; E) \simeq \mathrm{Fun}(BG, E)$$

Let $X \in Sp_G$, $K \leq G$, then

$$X^{hK} := \mathrm{Map}(BG, X)^K$$

-

$$\begin{tikzcd} \mathrm{Span}(\mathrm{Fin}_G^{free}) \arrow[r] \arrow[dr, "{\mathrm{Mack}(\mathrm{Fin}_G^{free} ; E) \cong \mathrm{Fun}(BG, E)}"'] & \mathrm{Span}(\mathrm{Fin}_G) \arrow[d, "{\mathrm{Mack}(\mathrm{Fin}_G ; E)}"] \\ & E \end{tikzcd}$$